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GRADED BETTI NUMBERS OF IDEALS WITH LINEAR QUOTIENTS

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In this paper we show that every ideal with linear quotients is componentwise linear. We also generalize the Eliahou-Kervaire formula for graded Betti numbers of stable ideals to homogeneous ideals with linear quotients.

1. Introduction

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and $I \subset S$ a homogeneous ideal. Let $\{f_1, \dots, f_m\}$ be a system of homogeneous generators for I . We say that I has *linear quotients with respect to the elements* f_1, \dots, f_m if the ideal $\langle f_1, \dots, f_{i-1} \rangle : f_i$ is generated by linear forms for all $i = 2, \dots, m$ (notice that this property depends on the order of the generators). Monomial ideals with linear quotients were introduced in [4] by Herzog and Takayama and have strong combinatorial implication, see the paper of Soleyman Jahan and Zheng [5].

In this paper we study the minimal free resolution of a graded ideal I with linear quotients with respect to a *minimal* system of homogeneous generators $\{f_1, \dots, f_m\}$ (in this case we simply say that I has *linear quotients*). It is known

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that I is componentwise linear provided that $\deg(f_1) \leq \dots \leq \deg(f_m)$, see the book of Herzog and Hibi [3]. We prove that I is componentwise linear without the additional assumption on the degrees, giving an affirmative answer to a question of Herzog (see Corollary 2.4). Our result is a generalization of [5, Corollary 2.8], where the authors prove componentwise linearity in case I is a monomial ideal and f_1, \dots, f_m are monomials.

A large class of monomial ideals with linear quotients is the class of stable ideals. A minimal free resolution of stable ideals was constructed by Eliahou and Kervaire in [1], who also give an explicit formula for their graded Betti numbers. For an ideal with linear quotients we give a formula for its Betti numbers that generalizes the formula by Eliahou and Kervaire. We express the graded Betti numbers in terms of the numbers $\deg(f_i)$ and the minimal number of generators of $\langle f_1, \dots, f_{i-1} \rangle : f_i$, for $i = 1, \dots, m$. As a consequence we will also obtain the Castelnuovo-Mumford regularity and the projective dimension of I .

2. Graded Ideals With Linear Quotients

Let $I \subset S$ be a graded ideal and $\{f_1, \dots, f_m\}$ be a minimal system of homogeneous generators for I such that I has linear quotients with respect to f_1, \dots, f_m . Notice that if an ideal has linear quotients with respect to a minimal system of homogeneous generators then it need not have linear quotients with respect to all minimal homogeneous system of generators (see Example 2.1).

It is easy to see that $\deg(f_i) \geq \min\{\deg(f_1), \dots, \deg(f_{i-1})\}$ for $i = 2, \dots, m$. Particularly, $\deg(f_1) \leq \deg(f_i)$ for every $i = 1, \dots, m$. But in general, the sequence $\deg(f_1), \dots, \deg(f_{i-1})$ of degrees need not be increasing.

For example, the ideal $\langle xy, xy^3z + y^4z - y^3z^2, x^3 + x^2y - x^2z, x^2z^3 \rangle$ has linear quotients in the given order, but $\deg(xy^3z + y^4z - y^3z^2) > \deg(x^3 + x^2y - x^2z)$.

Example 2.1. Set $\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subset S$ the maximal irrelevant ideal. It is easy to see that \mathfrak{m}^k has linear quotients with respect to its unique system of monomial generators for every natural number k (for example ordering the monomials lexicographically).

If K is large enough, we can choose a minimal system of homogeneous generators $\{g_1, \dots, g_t\}$ for \mathfrak{m}^k such that $\text{GCD}(g_i, g_j) = 1$ for each $i \neq j$. Hence it is clear that, if $k \geq 2$, \mathfrak{m}^k does not have linear quotients with respect to $\{g_1, \dots, g_t\}$, for any order, otherwise the unique factorization would be contradicted.

If J is a graded ideal of S , then we write $J_{\langle j \rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to J . Moreover, we write $J_{\leq k}$ for the ideal generated by all homogeneous polynomials in J whose degree is

less than or equal to k . We will say that J is componentwise linear if $J_{\langle j \rangle}$ has a j -linear resolution for all j .

In [5, Lemma 2.1] the authors show that for any monomial ideal J with linear quotients with respect to the unique minimal system $G(J)$ of monomial generators of J , there exists a degree increasing order of the elements u_1, \dots, u_m of $G(J)$ such that J has linear quotients with respect to this order. As a corollary they are able to prove that J is componentwise linear. It is worth remarking that if J has linear quotients with respect to a non-minimal system of generators, then J is not necessarily componentwise linear (see Example 2.5). We prove the result of [5] for an arbitrary graded ideal with linear quotients using completely different methods.

In the following for a \mathbb{Z} -graded module M we write $M(a)$ for the module obtained by *shifting degrees* by a , for any $a \in \mathbb{Z}$: i.e. $M(a)$ is the module M with the grading defined as $M(a)_b = M_{a+b}$.

For our purpose we need the next lemma.

Lemma 2.2. *Let $J = \langle f_1, \dots, f_m \rangle$ be a componentwise linear ideal and let $I = J + \langle f \rangle$ where f is a homogenous form of degree d . If the ideal $J : f$ is generated by linear forms then $I_{\langle j \rangle}$ has a j -linear resolution for each $j \leq d$ provided that f_1, \dots, f_m, f is a minimal system of generators for I .*

Proof. $I_{\langle j \rangle} = J_{\langle j \rangle}$ for each $j < d$ so it is enough to prove that $I_{\langle d \rangle}$ has d -linear resolution. First we show that $J : f = J_{\langle d \rangle} : f$. It is clear that $J_{\langle d \rangle} \subseteq J_{\leq d} \subseteq J$. Therefore $J_{\langle d \rangle} : f \subseteq J_{\leq d} : f \subseteq J : f$. Set $L = J : f = \langle l_1, \dots, l_r \rangle$ where l_i are linear forms. So there exist non-zero homogeneous polynomials $\lambda_{t_{1i}}, \dots, \lambda_{t_{s_i i}}$ such that $l_i f = \sum_{k=1}^{s_i} \lambda_{t_{ki}} f_{t_{ki}}$ and $\deg(\lambda_{t_{ki}}) + \deg(f_{t_{ki}}) = \deg(l_i f) = d + 1$. Since f_1, \dots, f_m, f is a minimal system of generators for I , $\deg(\lambda_{t_{ki}}) \geq 1$ and so $\deg(f_{t_{ki}}) \leq d$. Then $l_i f \in J_{\leq d}$ but it is clear that $(J_{\langle d \rangle})_{d+1} = (J_{\leq d})_{d+1}$. So $l_i f \in J_{\langle d \rangle}$ and $J : f = J_{\langle d \rangle} : f$.

Trivially $J_{\langle d \rangle} + \langle f \rangle = I_{\langle d \rangle}$, so by the above discussion

$$I_{\langle d \rangle} / J_{\langle d \rangle} \cong (S / (J_{\langle d \rangle} : f))(-d) = (S / (J : f))(-d) = S / L(-d).$$

Hence we can consider the short exact sequence

$$0 \rightarrow J_{\langle d \rangle} \rightarrow I_{\langle d \rangle} \rightarrow S / L(-d) \rightarrow 0$$

which yields the long exact sequence of Tor

$$\dots \rightarrow \text{Tor}_i^S(K, J_{\langle d \rangle})_{i+j} \rightarrow \text{Tor}_i^S(K, I_{\langle d \rangle})_{i+j} \rightarrow \text{Tor}_i^S(K, S / L(-d))_{i+j} \rightarrow \dots$$

The Tor-groups on the right and on the left end of this sequence vanish for $j \neq d$ since the corresponding modules have d -linear resolution. This proves that $I_{\langle d \rangle}$ has d -linear resolution. \square

We are ready to prove the main theorem of the paper.

Theorem 2.3. *Let $J = \langle f_1, \dots, f_m \rangle$ be a componentwise linear ideal and let $I = J + \langle f \rangle$ where f is a homogeneous form of degree d . If the ideal $J : f$ is generated by linear forms then I is componentwise linear provided that f_1, \dots, f_m, f is a minimal system of generators for I .*

Proof. By Lemma 2.2, $I_{\langle j \rangle}$ has j -linear resolution for each $j \leq d$. Set $\mathbf{m} = \langle x_1, \dots, x_n \rangle$ and $p = \max\{\deg(f_i)\}$. if $d > p$ then $I_{\langle d+j \rangle} = \mathbf{m}^j I_{\langle d \rangle}$ for each $j \geq 1$. So the exact sequence

$$0 \rightarrow I_{\langle d+i \rangle} \rightarrow I_{\langle d+i-1 \rangle} \rightarrow I_{\langle d+i-1 \rangle} / I_{\langle d+i \rangle} \rightarrow 0$$

and induction on i shows that $I_{\langle d+i \rangle}$ has $d+i$ -linear resolution. Therefore, I is componentwise linear ideal.

If $d \leq p$, let $s = p - d$. In this case we use induction on s to prove the result. If $s = 0$, again by Lemma 2.2, I is componentwise linear arguing as above. Now suppose that $s \geq 1$ and the result is true for $s - 1$. Let $L = J : f = \langle l_1, \dots, l_r \rangle$ minimally generated by r linear forms. If $r = n$ then $\mathbf{m}f \subseteq J$. So $J_{\langle d+i \rangle} = I_{\langle d+i \rangle}$ for each $i \geq 1$. Now another application of Lemma 2.2 shows that I is a componentwise linear ideal. If $r < n$ we complete $\{l_1, \dots, l_r\}$ to a K -basis of S_1 by new linear forms e_1, \dots, e_{n-r} . Set $g_i = e_i f$ for $i = 1, \dots, n - r$. One can easily check that $(J + \langle g_1, \dots, g_{n-r} \rangle)_{\langle d+i \rangle} = I_{\langle d+i \rangle}$ for each $i \geq 1$. Therefore it is enough to show that $J + \langle g_1, \dots, g_{n-r} \rangle$ is componentwise linear.

Claim 1: For every $i = 1, \dots, n - r$ we have $(J + \langle g_1, \dots, g_{i-1} \rangle) : g_i = L + \langle e_1, \dots, e_{i-1} \rangle$.

Proof of Claim 1: Let $h \in (J + \langle g_1, \dots, g_{i-1} \rangle) : g_i$. So $he_i f \in J + \langle g_1, \dots, g_{i-1} \rangle$. Therefore there exist suitable coefficients γ_j and λ_j such that $he_i f = \sum_{j=1}^m \gamma_j f_j + \sum_{j=1}^{i-1} \lambda_j e_j f$. Then $he_i - \sum_{j=1}^{i-1} \lambda_j e_j \in J : f$. So $he_i \in \langle l_1, \dots, l_r, e_1, \dots, e_{i-1} \rangle$. Since $l_1, \dots, l_r, e_1, \dots, e_{n-r}$ is a regular sequence $h \in L + \langle e_1, \dots, e_{i-1} \rangle$. The other inclusion is clear.

Claim 2: For every $i = 1, \dots, n - r$ the ideal $J + \langle g_1, \dots, g_i \rangle$ is minimally generated by $\{f_1, \dots, f_m, g_1, \dots, g_i\}$.

Proof of Claim 2: Suppose that for some t , $f_t \in \langle f_1, \dots, \widehat{f_t}, \dots, f_m, g_1, \dots, g_i \rangle$. Then there exist suitable coefficients γ_j and λ_j such that $f_t = \sum_{\substack{j=1 \\ j \neq t}}^m \gamma_j f_j + \sum_{j=1}^i \lambda_j g_j =$

$\sum_{\substack{j=1 \\ j \neq t}}^m \gamma_j f_j + (\sum_{j=1}^i \lambda_j e_j) f$. So $\{f_1, \dots, f_m, f\}$ is not a minimal system of generators for I which is a contradiction.

If for some t , $g_t \in \langle f_1, \dots, f_m, g_1, \dots, \widehat{g_t}, \dots, g_i \rangle$ then again one can find coefficients

γ_j and λ_j such that $g_t = \sum_{j=1}^m \gamma_j f_j + \sum_{\substack{j=1 \\ j \neq t}}^i \lambda_j g_j$. So $(e_t - \sum_{\substack{j=1 \\ j \neq t}}^i \lambda_j e_j)f \in J$ and therefore

$e_t - \sum_{\substack{j=1 \\ j \neq t}}^i \lambda_j e_j \in L$, which is a contradiction.

Since $\deg(g_i) = d + 1$, Claim 1 and Claim 2 together with the inductive hypothesis show that $J + \langle g_1, \dots, g_i \rangle$ is componentwise linear for each $i = 1, \dots, n - r$. This completes the proof. \square

From the above theorem the answer to a question by Herzog follows.

Corollary 2.4. *Let I be a homogeneous ideal and $\{f_1, \dots, f_m\}$ be a minimal system of generators for I . If I has linear quotients with respect to f_1, \dots, f_m then I is a componentwise linear ideal.*

Proof. Set $I_i = \langle f_1, \dots, f_i \rangle$ for $i = 1, \dots, m$. It is clear that I_1 has a linear resolution. In particular I_1 is a componentwise linear ideal. Therefore, one can apply Theorem 2.3 and induction on i to see that each I_i is componentwise linear for $i = 1, \dots, m$. \square

Example 2.5. In Corollary 2.4 one cannot remove the hypothesis that I is minimally generated by $\{f_1, \dots, f_m\}$. For instance, let I be the monomial ideal $I = \langle x^2, y^2 \rangle \subset K[x, y]$. Trivially $I = I_{\langle 2 \rangle}$ does not have a 2-linear resolution, but it has linear quotients with respect to the system of generators $\{x^2, xy^2, y^2\}$.

In general we can observe that every \mathfrak{m} -primary ideal $I \subset S$ ($\mathfrak{m} = \langle x_1, \dots, x_n \rangle$), has linear quotients with respect to a homogeneous system of generators. In fact let h be such that $\mathfrak{m}^h \subset I$. Then clearly by ordering the elements of \mathfrak{m}^h appropriately (for example lexicographically), I has linear quotients with respect to $\{\mathfrak{m}^h, I_{h-1}, \dots, I_s\}$, where s is minimal such that $I_s \neq 0$.

In the following we compute the graded Betti numbers of ideals with linear quotients. For this, we use the formula (1) of Herzog and Hibi (see [2, Proposition 1.3]) for the graded Betti numbers of an arbitrary componentwise linear ideal J .

$$\beta_{i,i+j}(J) = \beta_i(J_{\langle j \rangle}) - \beta_i(\mathfrak{m}J_{\langle j-1 \rangle}) \quad \text{for all } i, j \quad (1)$$

Theorem 2.6. *Let $J = \langle f_1, \dots, f_m \rangle$ be a componentwise linear ideal and let $I = J + \langle f \rangle$ where f is a homogeneous form of degree d . If the ideal $J : f$ is minimally generated by r linear forms then*

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(J) \quad \text{if } j \neq d,$$

$$\beta_{i,i+d}(I) = \beta_{i,i+d}(J) + \binom{r}{i},$$

provided that $\{f_1, \dots, f_m, f\}$ is a minimal system of homogeneous generators for I .

Proof. Consider the exact sequence

$$0 \rightarrow J \rightarrow I \rightarrow S/L(-d) \rightarrow 0 \quad (2)$$

which yields the long exact sequence

$$\begin{aligned} \dots \rightarrow \operatorname{Tor}_{i+1}^S(K, S/L(-d))_{i+j} &\rightarrow \operatorname{Tor}_i^S(K, J)_{i+j} \rightarrow \\ &\rightarrow \operatorname{Tor}_i^S(K, I)_{i+j} \rightarrow \operatorname{Tor}_i^S(K, S/L(-d))_{i+j} \rightarrow \end{aligned}$$

The Tor-groups on the right and on the left end of this sequence vanish for $j \neq d, d+1$ since the corresponding module has a d -linear resolution. This proves that $\beta_{i,i+j}(I) = \beta_{i,i+j}(J)$ if $j \neq d, d+1$.

First consider $j = d$. As discussed in the proof of Lemma 2.2, one has the short exact sequence

$$0 \rightarrow J_{\langle d \rangle} \rightarrow I_{\langle d \rangle} \rightarrow S/L(-d) \rightarrow 0$$

which yields the long exact sequence

$$\dots \rightarrow \operatorname{Tor}_i^S(K, J_{\langle d \rangle})_{i+j} \rightarrow \operatorname{Tor}_i^S(K, I_{\langle d \rangle})_{i+j} \rightarrow \operatorname{Tor}_i^S(K, S/L(-d))_{i+j} \rightarrow \dots$$

Since both $J_{\langle d \rangle}$ and $S/L(-d)$ have a d -linear resolution, the above long exact sequence gives

$$\beta_i(I_{\langle d \rangle}) = \beta_i(J_{\langle d \rangle}) + \beta_i(S/L(-d))$$

Moreover the minimal free resolution of $S/L(-d)$ is given by Koszul complex, so

$$\beta_i(S/L(-d)) = \binom{r}{i}$$

Thus by (1),

$$\begin{aligned} \beta_{i,i+d}(I) &= \beta_i(I_{\langle d \rangle}) - \beta_i(\mathfrak{m}I_{\langle d-1 \rangle}) \\ &= \beta_i(I_{\langle d \rangle}) - \beta_i(\mathfrak{m}J_{\langle d-1 \rangle}) \\ &= \beta_i(J_{\langle d \rangle}) + \binom{r}{i} - \beta_i(\mathfrak{m}J_{\langle d-1 \rangle}) \\ &= \beta_{i,i+d}(J) + \binom{r}{i}. \end{aligned}$$

Now consider $j = d + 1$. By (2) we have the following exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Tor}_{i+1}^S(K, J)_{i+d+1} &\rightarrow \operatorname{Tor}_{i+1}^S(K, I)_{i+d+1} \rightarrow \operatorname{Tor}_{i+1}^S(K, S/L(-d))_{i+d+1} \\ &\rightarrow \operatorname{Tor}_i^S(K, J)_{i+d+1} \rightarrow \operatorname{Tor}_i^S(K, I)_{i+d+1} \rightarrow 0. \end{aligned}$$

So by the first part $\beta_{i,i+d+1}(I) =$

$$\begin{aligned} &= \beta_{i,i+d+1}(J) - \binom{r}{i+1} + \beta_{i+1,i+d+1}(J) + \binom{r}{i+1} - \beta_{i+1,i+d+1}(J) \\ &= \beta_{i,i+d+1}(J). \end{aligned}$$

□

Corollary 2.7. *Let I be a homogeneous ideal with linear quotients with respect to f_1, \dots, f_m where $\{f_1, \dots, f_m\}$ is a minimal system of homogeneous generators for I . Let n_p be the minimal number of homogeneous generators of $\langle f_1, \dots, f_{p-1} \rangle : f_p$ for $p = 1, \dots, m$. Then*

$$\operatorname{reg}(I) = \max\{\deg(f_p) : p = 1, \dots, m\}$$

$$\operatorname{projdim}(I) = \max\{n_p | 1 \leq p \leq m\}$$

$$\beta_{i,i+j}(I) = \sum_{1 \leq p \leq m, \deg(f_p)=j} \binom{n_p}{i}$$

$$\beta_i(I) = \sum_{p=1}^m \binom{n_p}{i}.$$

Proof. It suffices to compute the graded Betti numbers of I . Let $I_p = \langle f_1, \dots, f_p \rangle$. Using Corollary 2.4 and Theorem 2.6 it is easy to find the Betti numbers of each I_p with inductive process on p . □

A stable ideal is a monomial ideal $J \subset S = K[x_1, \dots, x_n]$ with the following property: if u is a monomial belonging to J , $m(u) := \max\{k : x_k | u\}$ and $i < m(u)$, then $x_i(u/x_{m(u)}) \in J$.

Suppose that J is a stable ideal and the minimal monomial generators u_1, \dots, u_m of J are ordered so that $i < q$ if and only if either the degree of u_i is less than the degree of u_q , or the degrees are equal and $u_i >_{\text{revlex}} u_q$. One can easily see that for $p \geq 1$ we have

$$\langle u_1, \dots, u_{p-1} \rangle : u_p = \langle x_1, \dots, x_{m(u_p)-1} \rangle.$$

In particular J has linear quotients.

The minimal free resolution of a stable ideal is given by the Eliahou-Kervaire

resolution [1]. From this resolution one can easily find the regularity, the projective dimension and the Betti numbers of a stable ideal. The Eliahou-Kervaire formula for the graded Betti numbers of a stable ideal J as above is the following:

$$\beta_{i,i+j}(J) = \sum_{1 \leq p \leq m, \deg(u_p)=j} \binom{m(u_p)-1}{i}$$

From the above discussion it is clear that Corollary 2.7 generalizes the Eliahou-Kervaire formula to all the homogenous ideals with linear quotients.

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